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K-Even Edge-Graceful Labeling of Some Cycle Related Graphs

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ABSTRACT: *In 1985, Lo[6] introduced the notion of edge-graceful graphs. In [4], Gayathri et al., introduced the even edge-graceful graphs. In [8], Sin-Min Lee, Kuo-Jye Chen and Yung-Chin Wang introduced the k-edge-graceful graphs.We introduced k-even edge-graceful graphs. In this paper, we investigate the k-even edge-gracefulness of some cycle related graphs.*

KEYWORDS: *k*-even edge-graceful labeling, *k*-even edge-graceful graphs. AMS(MOS) subject classification: 05*C*78.

I. INTRODUCTION

All graphs in this paper are finite, simple and undirected. Terms not defined here are used in the sense of Harary[5]. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. The cardinality of the vertex set is called the order of G denoted by p. The cardinality of the edge set is called the size of G denoted by q. A graph with p vertices and q edges is called a (p, q) graph.

In 1985, Lo[6] introduced the notion of edge-graceful graphs. In [4], Gayathri et al., introduced the even edge-graceful graphs and further studied in. In [8], Sin-Min Lee, Kuo-Jye Chen and Yung-Chin Wang introduced the *k*-edge-graceful graphs. We have introduced *k*-even edge- graceful graphs.

Definition 1.1:

k-even edge-graceful labeling (*k-EEGL*) of a (p, q) graph $G(V, E)$ is an injection f from E to $\{2k – 1, 2k, 2k + 1, ..., 2k + 2q – 2\}$ such that the induced mapping f^+ defined on V by $f^+(x) = (\Sigma f(xy)) \pmod{2s}$ taken over all edges xy are distinct and even where $s = max\{p, q\}$ and k is an integer greater than or equal to 1. A graph G that admits *k*-even edge-graceful labeling is called a *k*-even edge-graceful graph (*K-EEGG*).

Remark 1.2:

1-even edge-graceful labeling is an even edge-graceful labeling.

The definition of *k*-edge-graceful and *k*-even edge-graceful are equivalent to one another in the case of trees.

The edge-gracefulness and even edge-gracefulness of odd order trees are still open. The theory of 1-even edge-graceful is completely different from that of *k*- even edge-graceful. For example, tree of order 4 is 2-even edge-graceful but not 1-even edge-graceful. In this paper we investigate the *k*-even edge-gracefulness of some cycle related graphs. Throughout this paper, we assume that *k* is a positive integer greater than or equal to 1.

2. Prior Results:

1.Theorem : If a (p, q) graph G is *k*-even edge-graceful with all edges labeled with even numbers and $p \geq q$ then G is *k*-edge-graceful.

2.Theorem : If a (p, q) graph G is *k*-even edge-graceful in which all edges are labeled with even numbers and

$p \geq q$ then $q(q + 2k - 1) \equiv \frac{p(p+1)}{2} \pmod{p}$.

Further $$q(q + 2k - 1) \equiv \begin{cases} 0 \pmod{p} & \text{if } p \text{ is odd} \\ \frac{p}{2} \pmod{p} & \text{if } p \text{ is even} \end{cases}$$

3.Theorem : If a (p, q) graph G is k-even edge-graceful in which all edges are labeled with even numbers and $p \geq q$ then $p \equiv 0$, 1 or 3 (mod 4).

4.Theorem : If a (p, q) graph G is a k-even edge-graceful tree of odd order then

$$k = \frac{p}{2}(l - 1) + 1$$

where l is any odd positive integer and hence $k \equiv 1 (\text{mod } p)$.

5. Observation : We observe that any tree of odd order p has the sum of the labels congruent to 0 (mod p).

6. Theorem : If a (p, q) graph G is a k-even edge-graceful tree of even order with $p \equiv 0 \pmod 4$ then $k = \frac{p}{4}(2l - 1) + 1$ where l is any positive integer.

Further $$k \equiv \begin{cases} \frac{p+4}{4} \pmod p & \text{if } l \text{ is odd} \\ \frac{3p+4}{4} \pmod p & \text{if } l \text{ is even} \end{cases}$$

2. MAIN RESULTS

Definition 2.1

Let C_n denote the cycle of length n. Then the join of $\{e\}$ with any one vertex of C_n is denoted by $C_n \cup \{e\}$. In this graph, $p = q = n + 1$.

Theorem 2.2

The graph $C_n \cup \{e\}$ of even order is k-even-edge-graceful for all $k \equiv z \left(\text{mod } \frac{p}{2}\right)$, where $0 \leq z \leq \frac{p}{2} - 1$, $n \equiv 1 \pmod 4$ and $n \neq 1$.

Proof

Let $\{v_1, v_2, ..., v_n, v\}$ be the vertices of $C_n \cup \{e\}$, the edges $e_i = (v_i, v_{i+1})$ for $1 \leq i \leq n-1$; $e_n = (v_n, v_1)$ and $e_{n+1} = (v_2, v)$ (see Figure 1).

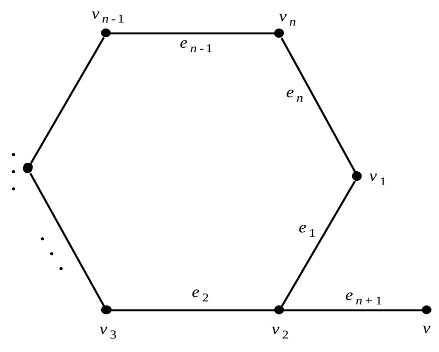

Figure 1: $C_n \cup \{e\}$ with ordinary labeling

First, we label the edges as follows:

For $k \geq 1$, $1 \leq i \leq n$ and i is odd,

$$f(e_i) = 2k + i - 2.$$

When i is even, we label the edges as follows:

For $k \equiv z\left(\text{mod } \frac{p}{2}\right)$, $0 \leq z \leq \frac{p-2}{4}$,

$$f(e_i) = \begin{cases} 2k + n + i - 2 & \text{for } 1 \le i < \dfrac{n - 4z + 7}{2} \\ 2k + n + i & \text{for } \dfrac{n - 4z + 7}{2} \le i \le n. \end{cases}$$

For $k \equiv \frac{p+2}{4}\left(\bmod \frac{p}{2}\right)$,

$$f(e_i) = 2k + n + i.$$

For $k \equiv \frac{p+6}{4}\left(\bmod \frac{p}{2}\right)$,

$$f(e_i) = 2k + n + i - 2.$$

For $k \equiv z\left(\bmod \frac{p}{2}\right)$, $\frac{p+10}{4} \le z \le \frac{p}{2} - 1$,

$$f(e_i) = \begin{cases} 2k + n + i - 2 & \text{for } 1 \le i < \dfrac{3n - 4z + 9}{2} \\ 2k + n + i & \text{for } \dfrac{3n - 4z + 9}{2} \le i \le n. \end{cases}$$

$$f(e_{n+1}) = \begin{cases} 2k & \text{when } k \equiv 0 \pmod p \\ 2k + 2n - 2z + 2 & \text{when } k \equiv z \pmod p \text{ and } 1 \le z \le p - 1. \end{cases}$$

Then the induced vertex labels are as follows:

Case 1: $\boldsymbol{k \equiv z\left(\bmod \frac{p}{2}\right),\ 0 \le z \le \frac{p-2}{4}}$

$$f^+(v_i) = \begin{cases} n + 4z + 2i - 5 & \text{for } 1 \le i < \dfrac{n - 4z + 7}{2} \\ 4z - n + 2i - 5 & \text{for } \dfrac{n - 4z + 7}{2} \le i \le n. \end{cases}$$

Case 2: $\boldsymbol{k \equiv \frac{p+2}{4}\left(\bmod \frac{p}{2}\right)}$

$f^+(v_1) = 2n$; $f^+(v_i) = 2i - 2$ for $2 \le i \le n$.

Case 3: $\boldsymbol{k \equiv \frac{p+6}{4}\left(\bmod \frac{p}{2}\right)}$

$f^+(v_i) = 2i$ for $1 \le i \le n$.

Case 4: $\boldsymbol{k \equiv z\left(\bmod \frac{p}{2}\right), \frac{p+10}{4} \le z \le \frac{p}{2} - 1}$

$$f^{+}(v_i) = \begin{cases} 4z - n + 2i - 7 & \text{for } 1 \le i < \dfrac{3n - 4z + 9}{2} \\ 4z - 3n + 2i - 7 & \text{for } \dfrac{3n - 4z + 9}{2} \le i \le n. \end{cases}$$

For $k \equiv z \left(\bmod \dfrac{p}{2} \right)$, $0 \le z \le \dfrac{p}{2} - 1$,

$f^{+}(v_{n+1}) = 0.$

Therefore, $f^{+}(V) = \{0, 2, 4, ..., 2s - 2\}$, where $s = \max\{p, q\} = n + 1$. So, it follows that the vertex labels are all distinct and even. Hence, the graph $C_n \cup \{e\}$ of even order is k-even-edge-graceful for all $k \equiv z \left(\bmod \dfrac{p}{2} \right)$, where $0 \le z \le \dfrac{p}{2} - 1$, $n \equiv 1 \pmod 4$ and $n \ne 1$. ■

For example, consider the graph $C_{13} \cup \{e\}$. Here $n = 13$;

$s = \max\{p, q\} = 14$; $2s = 28$. The 21-*EEGL* of $C_{13} \cup \{e\}$ is given in Figure 2.

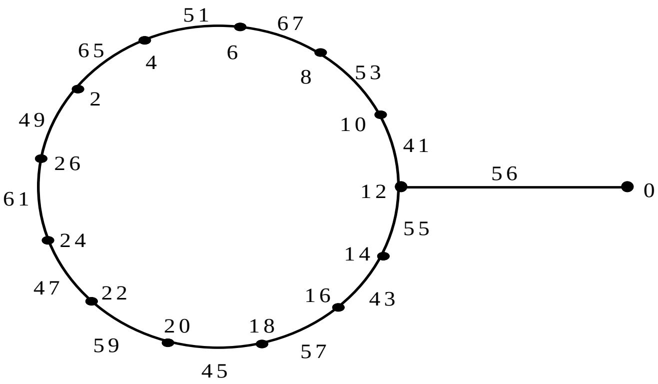

Figure 2: 21-*EEGL* of $C_{13} \cup \{e\}$

For example, consider the graph $C_{17} \cup \{e\}$. Here $n = 17$;

$s = \max\{p, q\} = 18$; $2s = 36$. The 5-*EEGL* of $C_{17} \cup \{e\}$ is given in Figure 3.

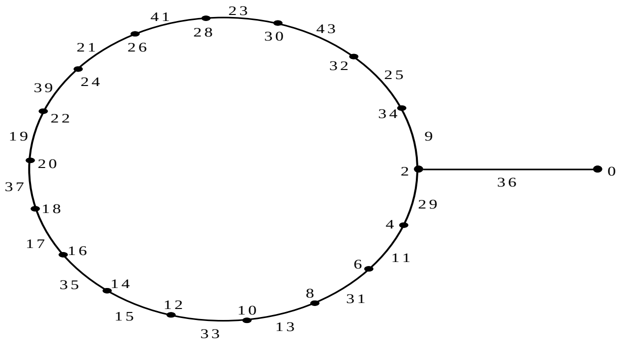

Figure 3: 5-*EEGL* of $C_{17} \cup \{e\}$

Definition 2.3 [8]

For $p \geq 4$, a cycle (of order p) with one chord is a simple graph obtained from a p-cycle by adding a chord. Let the p-cycle be $v_1v_2 \ldots v_pv_1$. Without loss of generality, we assume that the chord joins v_1 with any one v_i, where $3 \leq i \leq p - 1$. This graph is denoted by $C_p(i)$. For example $C_p(5)$ means a graph obtained from a p-cycle by adding a chord between the vertices v_1 and v_5. In this graph, $q = p + 1$.

Theorem 2.4

The graph $C_n(i)$, $(n > 4)$, $3 \leq i \leq n - 1$, cycle with one chord of odd order is k-even-edge-graceful for all $k \equiv z\left(\bmod \frac{q}{2}\right)$, where $0 \leq z \leq \frac{q}{2} - 1$.

Proof

Let $\{v_1, v_2, v_3, \ldots, v_i, v_{i+1}, \ldots, v_n\}$ be the vertices of $C_n(i)$, the edges $e_i = (v_i, v_{i+1})$ for $1 \leq i \leq n - 1$; $e_n = (v_n, v_1)$ and $e_{n+1} = (v_1, v_i)$, $3 \leq i \leq n - 1$ (see Figure 4). The chord connecting the vertex v_1 with v_i, $(i \geq 3)$ is shown in Figure 4. For this graph, $p = n$ and $q = n + 1$.

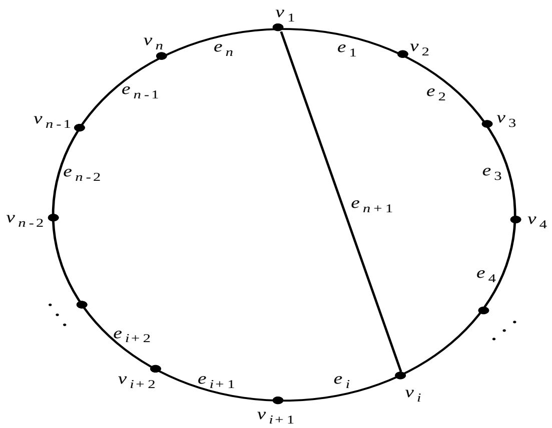

Figure 4: $C_n(i)$ with ordinary labeling

First, we label the edges as follows:
For $k \geq 1$ and $1 \leq i \leq n$,

$$f(e_i) = \begin{cases} 2k + i - 2 & \text{when } i \text{ is odd} \\ 2k + n + i - 2 & \text{when } i \text{ is even.} \end{cases}$$

$$f(e_{n+1}) = \begin{cases} 2k & \text{when } k \equiv 0 \pmod{q} \\ 2k + 2n - 2z + 2 & \text{when } k \equiv z \pmod{q}, 1 \leq z \leq q - 1. \end{cases}$$

Then the induced vertex labels are as follows:

Case I: $n \equiv 1 \pmod 4$ and $n \neq 1$

Case 1: $k \equiv z\left(\bmod \frac{q}{2}\right)$, $0 \leq z \leq \frac{q+2}{4}$

$$f^+(v_i) = \begin{cases} 4z + n + 2i - 5 & \text{for } 1 \leq i < \frac{n - 4z + 7}{2} \\ 4z - n + 2i - 7 & \text{for } \frac{n - 4z + 7}{2} \leq i \leq n. \end{cases}$$

Case 2: $k \equiv \frac{q+6}{4}\left(\bmod \frac{q}{2}\right)$

$f^{+}(v_i) = 2i$ for $1 \le i \le n$.

Case 3: $k \equiv z\left(\bmod \frac{q}{2}\right)$, $\frac{q+10}{4} \le z \le \frac{q}{2} - 1$

$$f^{+}(v_i) = \begin{cases} 4z - n + 2i - 7 & \text{for } 1 \le i < \frac{3n - 4z + 9}{2} \\ 4z - 3n + 2i - 9 & \text{for } \frac{3n - 4z + 9}{2} \le i \le n. \end{cases}$$

Case II: $n \equiv 3 \pmod 4$ and $n \neq 3$

Case 1: $k \equiv z\left(\bmod \frac{q}{2}\right)$, $0 \le z \le \frac{q}{4}$

$$f^{+}(v_i) = \begin{cases} 4z + n + 2i - 5 & \text{for } 1 \le i < \frac{n - 4z + 7}{2} \\ 4z - n + 2i - 7 & \text{for } \frac{n - 4z + 7}{2} \le i \le n. \end{cases}$$

Case 2: $k \equiv \frac{q+4}{4}\left(\bmod \frac{q}{2}\right)$

$f^{+}(v_i) = 2i - 2$ for $1 \le i \le n$.

Case 3: $k \equiv z\left(\bmod \frac{q}{2}\right)$, $\frac{q+8}{4} \le z \le \frac{q}{2} - 1$

$$f^{+}(v_i) = \begin{cases} 4z - n + 2i - 7 & \text{for } 1 \le i < \frac{3n - 4z + 9}{2} \\ 4z - 3n + 2i - 9 & \text{for } \frac{3n - 4z + 9}{2} \le i \le n. \end{cases}$$

Therefore, $f^{+}(V) \subseteq \{0, 2, 4, ..., 2s - 2\}$, where $s = \max\{p, q\} = n + 1$. So, it follows that the vertex labels are all distinct and even. Hence, the graph

$C_n(i)$, $(n > 4)$, $3 \le i \le n - 1$, cycle with one chord of odd order is k-even-edge-graceful for all $k \equiv z\left(\bmod \frac{q}{2}\right)$,

$0 \le z \le \frac{q}{2} - 1$. ■

For example, consider the graph $C_9(5)$. Here $n = 9$; $s = \max\{p, q\} = 10$; $2s = 20$.

The *15*-even-edge-graceful labeling of $C_9(5)$ is given in Figure 5.

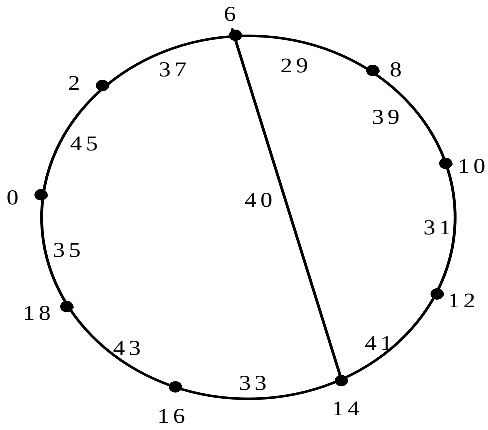

Figure 5: 15-*EEGL* of $C_9(5)$

For example, consider the graph $C_{11}(6)$. Here $n = 11$; $s = \max\{p, q\} = 12$; $2s = 24$. The *11-EEGL* of $C_{11}(6)$ is given in Figure 6.

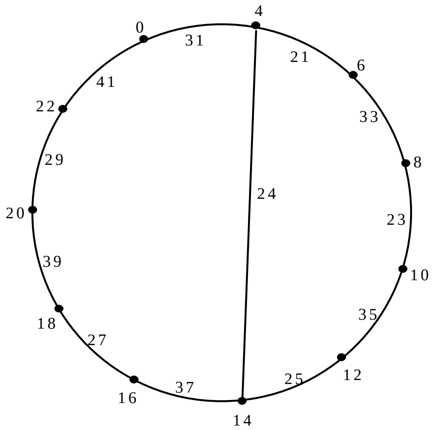

Figure 6: 11-*EEGL* of $C_{11}(6)$

Theorem 2.5

The graph $C_n(i)$, $(n \geq 4)$, $3 \leq i \leq n-1$, cycle with one chord of even order is *k*-even-edge-graceful for all $k \equiv z \pmod{q}$, where $0 \leq z \leq q-1$ and $n \equiv 0 \pmod{4}$.

Proof

Let the vertices and edges be defined as in Theorem 6.3.2.
First, we label the edges as follows:
For $k \geq 1$,

$$f(e_1) = 2k+1 \quad ; \quad f(e_2) = 2k-1$$

$$f(e_i) = 2k+2i-3 \qquad \text{for } 3 \leq i < \frac{n}{2}+3.$$

For $k \geq 1$ and $\frac{n}{2}+3 \leq i \leq n$,

$$f(e_i) = \begin{cases} 2k+2i+1 & \text{when } i \text{ is odd} \\ 2k+2i-3 & \text{when } i \text{ is even.} \end{cases}$$

$$f(e_{n+1}) = \begin{cases} 2k & \text{when } k \equiv 0 \pmod q \\ 2k + 2n - 2z + 2 & \text{when } k \equiv z \pmod q,\ 1 \le z \le q-1. \end{cases}$$

Then the induced vertex labels are as follows:

Case 1: $\boldsymbol{k \equiv 0 \pmod q}$

$$f^+(v_1) = 2n - 2 \quad ; \quad f^+(v_2) = 0.$$

$$f^+(v_3) = 2.$$

$$f^+(v_i) = \begin{cases} 4i - 8 & \text{for } 4 \le i < \frac{n}{2} + 3 \\ 4i - 2n - 6 & \text{for } \frac{n}{2} + 3 \le i \le n. \end{cases}$$

Case 2: $\boldsymbol{k \equiv z \pmod q,\ 1 \le z \le \frac{q-1}{2}}$

$$f^+(v_i) = 4z + 4i - 8 \quad \text{when } i = 1, 2.$$

For $k \equiv z \pmod q$, $1 \le z \le \frac{q-3}{2}$,

$$f^+(v_3) = 4z + 2$$

For $4 \le i \le n$,

$$f^+(v_i) = \begin{cases} 4z + 4i - 8 & \text{for } 4 \le i < \frac{n}{2} - z + 3 \\ 4z + 4i - 2n - 10 & \text{for } \frac{n}{2} - z + 3 \le i < \frac{n}{2} + 3 \\ 4z + 4i - 2n - 6 & \text{for } \frac{n}{2} + 3 \le i < n - z + 2 \\ 4z + 4i - 4n - 8 & \text{for } n - z + 2 \le i \le n. \end{cases}$$

For $k \equiv \frac{q-1}{2} \pmod q$,

$$f^+(v_3) = 0$$

$$f^+(v_i) = \begin{cases} 4i - 10 & \text{for } 4 \le i < \frac{n}{2} + 3 \\ 4i - 2n - 8 & \text{for } \frac{n}{2} + 3 \le i \le n. \end{cases}$$

Case 3: $\boldsymbol{k \equiv \frac{q+1}{2} \pmod q}$

$$f^+(v_1) = 2n \quad ; \quad f^+(v_2) = 2.$$

$$f^+(v_3) = 4.$$

$$f^+(v_i) = \begin{cases} 4i - 6 & \text{for } 4 \le i < \frac{n}{2} + 2 \\ 0 & \text{when } i = \frac{n}{2} + 2 \\ 4i - 2n - 4 & \text{for } \frac{n}{2} + 3 \le i \le n. \end{cases}$$

Case 4: $\boldsymbol{k \equiv z \,(\mathrm{mod}\; q),\ \frac{q+3}{2} \le z \le q - 1}$

$f^+(v_i) = 4z + 4i - 2n - 10$ when $i = 1, 2$.

$f^+(v_3) = 4z - 2n$.

For $k \equiv z$ (mod q), $\frac{q+3}{2} \le z \le q - 2$,

$$f^+(v_i) = \begin{cases} 4z + 4i - 2n - 10 & \text{for } 4 \le i < n - z + 3 \\ 4z + 4i - 4n - 12 & \text{for } n - z + 3 \le i < \frac{n}{2} + 3 \\ 4z + 4i - 4n - 8 & \text{for } \frac{n}{2} + 3 \le i < \frac{3n}{2} - z + 3 \\ 4z + 4i - 6n - 10 & \text{for } \frac{3n}{2} - z + 3 \le i \le n. \end{cases}$$

For $k \equiv q - 1$ (mod q),

$$f^+(v_i) = \begin{cases} 4i - 12 & \text{for } 4 \le i < \frac{n}{2} + 3 \\ 4i - 2n - 10 & \text{for } \frac{n}{2} + 3 \le i \le n. \end{cases}$$

Therefore, $f^+(V) \subseteq \{0, 2, 4, \ldots, 2s - 2\}$, where $s = \max\{p, q\} = n + 1$. So, it follows that the vertex labels are all distinct and even. Hence, the graph $C_n(i)$, ($n \ge 4$), $3 \le i \le n - 1$, cycle will one chord of even order is k-even-edge-graceful for all $k \equiv z$ (mod q), where $0 \le z \le q - 1$ and $n \equiv 0$ (mod 4). ■

For example, consider the graph $C_{16}(5)$, Here $n = 16$; $s = \max\{p, q\} = 17$; $2s = 34$.

The *18-EEGL* of $C_{16}(5)$ is given in Figure 7.

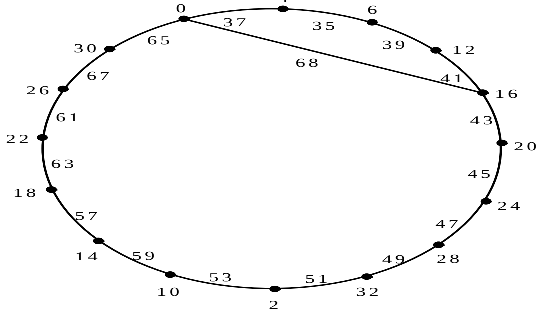

Figure 7: 18-*EEGL* of C_{16}(5)

Definition 2. 6

The crown $C_n \odot K_1$ is the graph obtained from the cycle C_n by attaching pendant edge at each vertex of the cycle and is denoted by C_n^+. In this graph, $p = q = 2n$.

Theorem 2.7

The crown graph C_n^+, $(n \geq 3)$ of even order is k-even-edge-graceful for all $k \equiv z\left(\bmod \dfrac{p}{3}\right)$, where $0 \leq z \leq \dfrac{p}{3} - 1$ and $n \equiv 0 \pmod 3$.

Proof

For this graph, $p = q = 2n$. Let $v_1, v_2, v_3, ..., v_n$ and $v_1', v_2', v_3', \ldots, v_n'$ be the vertices and pendant vertices of C_n^+ respectively.

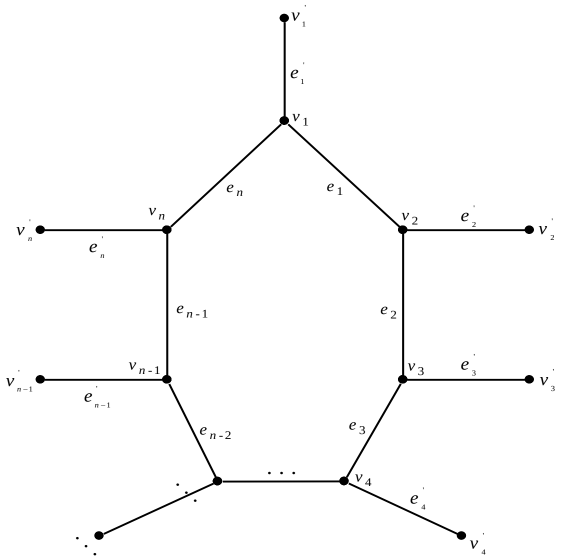

Figure 8: C_n^+ with ordinary labeling

The edges are defined by

$e_i = (v_i, v_{i+1})$ for $1 \leq i \leq n-1$; $e_n = (v_n, v_1)$

and $e_i' = (v_i, v_i')$ for $1 \leq i \leq n$ (see Figure 8).

First, we label the edges as follows:

For $k \geq 1$,

$f(e_i) = 2k + 4i - 5$ for $1 \leq i \leq n$

$f(e_1') = 2k$

$f(e_i') = 2k + 4(n - i + 1)$ for $2 \leq i \leq n$.

Then the induced vertex labels are as follows:

Case 1: $k \equiv 0\left(\bmod \dfrac{p}{3}\right)$

$$f^{+}(v_i) = \begin{cases} 4n+4i-10 & \text{when } i = 1,2 \\ 4i-10 & \text{for } 3 \le i \le n. \end{cases}$$

Case 2: $k \equiv z\left(\bmod \dfrac{p}{3}\right), 1 \le z \le \dfrac{p}{3} - 1$ **and z is odd**

$$f^{+}(v_i) = \begin{cases} 6z+4i-10 & \text{for } 1 \le i < n + \dfrac{5-3z}{2} \\ 6z-4n+4i-10 & \text{for } n + \dfrac{5-3z}{2} \le i \le n. \end{cases}$$

Case 3: $k \equiv z\left(\bmod \dfrac{p}{3}\right), 1 \le z \le \dfrac{p}{3} - 1$ **and z is even**

$$f^{+}(v_i) = \begin{cases} 6z+4i-10 & \text{for } 1 \le i < n + \dfrac{6-3z}{2} \\ 6z-4n+4i-10 & \text{for } n + \dfrac{6-3z}{2} \le i \le n. \end{cases}$$

The pendant vertices will have the labels (mod 2p) of the edges incident on them. Clearly, the vertex labels are all distinct and even. Hence, the crown graph C_n^{+}, $(n \ge 3)$ of even order is k-even-edge-graceful for all $k \equiv z\left(\bmod \dfrac{p}{3}\right)$, where $0 \le z \le \dfrac{p}{3} - 1$ and $n \equiv 0$ (mod 3). ■

For example, consider the graph C_6^{+}. Here $p = q = 12$; $s = \max\{p, q\} = 12$; $2s = 24$. The 2-even-edge-graceful labeling of C_6^{+} is given in Figure 9.

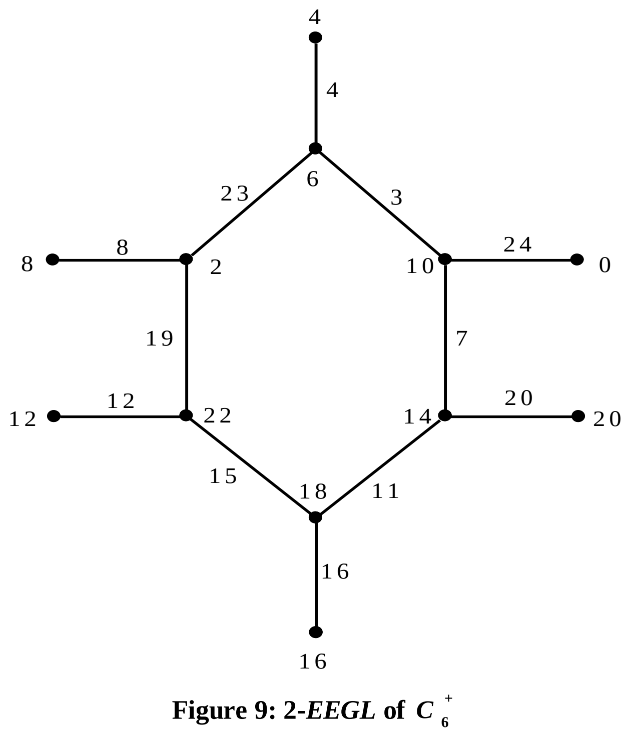

Figure 9: 2-*EEGL* of C_6^{+}

Theorem 2.8

The crown graph C_n^+, $(n \geq 4)$ of even order is k-even-edge-graceful for all $k \equiv z \pmod{p}$, where $0 \leq z \leq p - 1$, $n \equiv 1 \pmod 3$ and $n \neq 1$.

Proof

Let the vertices and edges be defined as in Theorem 6.4.2. The edge labels are also same as in Theorem 6.4.2.

Then the induced vertex labels are as follows:

Case 1: $\boldsymbol{k \equiv 0 \pmod{p}}$

$$f^+(v_i) = \begin{cases} 4n + 4i - 10 & \text{when } i = 1, 2 \\ 4i - 10 & \text{for } 3 \leq i \leq n. \end{cases}$$

When z is odd, the induced vertex labels are given below:

Case 2: $\boldsymbol{k \equiv z \pmod{p},\ 1 \leq z \leq \frac{p+1}{3}}$

$$f^+(v_i) = \begin{cases} 6z + 4i - 10 & \text{for } 1 \leq i < n + \frac{5 - 3z}{2} \\ 6z - 4n + 4i - 10 & \text{for } n + \frac{5 - 3z}{2} \leq i \leq n. \end{cases}$$

Case 3: $\boldsymbol{k \equiv z \pmod{p},\ \frac{p+4}{3} \leq z \leq \frac{2p+2}{3}}$

$$f^+(v_i) = \begin{cases} 6z - 4n + 4i - 10 & \text{for } 1 \leq i < 2n + \frac{5 - 3z}{2} \\ 6z - 8n + 4i - 10 & \text{for } 2n + \frac{5 - 3z}{2} \leq i \leq n. \end{cases}$$

Case 4: $\boldsymbol{k \equiv z \pmod{p},\ \frac{2p+5}{3} \leq z \leq p - 1}$

$$f^+(v_i) = \begin{cases} 6z - 8n + 4i - 10 & \text{for } 1 \leq i < 3n + \frac{5 - 3z}{2} \\ 6z - 12n + 4i - 10 & \text{for } 3n + \frac{5 - 3z}{2} \leq i \leq n. \end{cases}$$

When z is even, the induced vertex labels are given below:

Case 5: $\boldsymbol{k \equiv z \pmod{p},\ 1 \leq z \leq \frac{p+1}{3}}$

$$f^+(v_i) = \begin{cases} 6z + 4i - 10 & \text{for } 1 \leq i < n + \frac{6 - 3z}{2} \\ 6z - 4n + 4i - 10 & \text{for } n + \frac{6 - 3z}{2} \leq i \leq n. \end{cases}$$

Case 6: $\boldsymbol{k \equiv z \pmod{p},\ \frac{p+4}{3} \leq z \leq \frac{2p+2}{3}}$

$$f^{+}(v_i) = \begin{cases} 6z - 4n + 4i - 10 & \text{for } 1 \le i < 2n + \dfrac{6-3z}{2} \\ 6z - 8n + 4i - 10 & \text{for } 2n + \dfrac{6-3z}{2} \le i \le n. \end{cases}$$

Case 7: $\boldsymbol{k \equiv z \pmod{p}, \dfrac{2p+5}{3} \le z \le p-1}$

$$f^{+}(v_i) = \begin{cases} 6z - 8n + 4i - 10 & \text{for } 1 \le i < 3n + \dfrac{6-3z}{2} \\ 6z - 12n + 4i - 10 & \text{for } 3n + \dfrac{6-3z}{2} \le i \le n. \end{cases}$$

The pendant vertices will have the labels (mod $2p$) of the edges incident on them. Clearly, the vertex labels are all distinct and even. Hence, the crown graph C_n^+, $(n \ge 4)$ of even order is k-even-edge-graceful for all $k \equiv z \pmod{p}$, where $0 \le z \le p - 1$, $n \equiv 1 \pmod 3$ and $n \ne 1$. ■

For example, consider the graph C_7^+. Here $p = q = 14$; $s = \max\{p, q\} = 14$; $2s = 28$. The 5-even-edge-graceful labeling of C_7^+ is given in Figure 10.

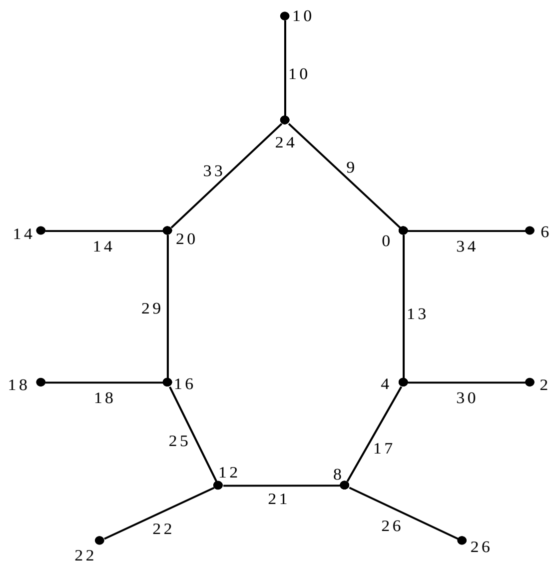

Figure 10: 5-*EEGL* of C_7^+

Theorem 2.9

The crown graph C_n^+, $(n \ge 5)$ of even order is k-even-edge-graceful for all $k \equiv z \pmod{p}$, where $0 \le z \le p - 1$, $n \equiv 2 \pmod 3$ and $n \ne 2$.

Proof

Let the vertices and edges be defined as in Theorem 2.7. The edge labels are also same as in Theorem 2.7.
Then the induced vertex labels are as follows:

Case 1: $\boldsymbol{k \equiv 0 \pmod{p}}$

$$f^{+}(v_i) = \begin{cases} 4n + 4i - 10 & \text{when } i = 1, 2 \\ 4i - 10 & \text{for } 3 \le i \le n. \end{cases}$$

When z is odd, the induced vertex labels are given below:

Case 2: $\boldsymbol{k \equiv z \ (\mathrm{mod}\ p),\ 1 \le z \le \frac{p+2}{3}}$

$$f^{+}(v_i) = \begin{cases} 6z + 4i - 10 & \text{for } 1 \le i < n + \frac{5-3z}{2} \\ 6z - 4n + 4i - 10 & \text{for } n + \frac{5-3z}{2} \le i \le n. \end{cases}$$

Case 3: $\boldsymbol{k \equiv z \ (\mathrm{mod}\ p),\ \frac{p+5}{3} \le z \le \frac{2p+1}{3}}$

$$f^{+}(v_i) = \begin{cases} 6z - 4n + 4i - 10 & \text{for } 1 \le i < 2n + \frac{5-3z}{2} \\ 6z - 8n + 4i - 10 & \text{for } 2n + \frac{5-3z}{2} \le i \le n. \end{cases}$$

Case 4: $\boldsymbol{k \equiv z \ (\mathrm{mod}\ p),\ \frac{2p+4}{3} \le z \le p - 1}$

$$f^{+}(v_i) = \begin{cases} 6z - 8n + 4i - 10 & \text{for } 1 \le i < 3n + \frac{5-3z}{2} \\ 6z - 12n + 4i - 10 & \text{for } 3n + \frac{5-3z}{2} \le i \le n. \end{cases}$$

When z is even, the induced vertex labels are given below:

Case 5: $\boldsymbol{k \equiv z \ (\mathrm{mod}\ p),\ 1 \le z \le \frac{p+2}{3}}$

$$f^{+}(v_i) = \begin{cases} 6z + 4i - 10 & \text{for } 1 \le i < n + \frac{6-3z}{2} \\ 6z - 4n + 4i - 10 & \text{for } n + \frac{6-3z}{2} \le i \le n. \end{cases}$$

Case 6: $\boldsymbol{k \equiv z \ (\mathrm{mod}\ p),\ \frac{p+5}{3} \le z \le \frac{2p+1}{3}}$

$$f^{+}(v_i) = \begin{cases} 6z - 4n + 4i - 10 & \text{for } 1 \le i < 2n + \frac{6-3z}{2} \\ 6z - 8n + 4i - 10 & \text{for } 2n + \frac{6-3z}{2} \le i \le n. \end{cases}$$

Case 7: $\boldsymbol{k \equiv z \ (\mathrm{mod}\ p),\ \frac{2p+4}{3} \le z \le p - 1}$

$$f^{+}(v_i) = \begin{cases} 6z - 8n + 4i - 10 & \text{for } 1 \le i < 3n + \frac{6-3z}{2} \\ 6z - 12n + 4i - 10 & \text{for } 3n + \frac{6-3z}{2} \le i \le n. \end{cases}$$

The pendant vertices will have the labels (mod 2p) of the edges incident on them. Clearly, the vertex labels are all distinct and even. Hence, the crown graph C_n^+, ($n \geq 5$) of even order is k-even-edge-graceful for all $k \equiv z \pmod{p}$, where $0 \leq z \leq p - 1$, $n \equiv 2 \pmod{3}$ and $n \neq 2$. ■

For example, consider the graph C_5^+. Here $p = q = 10$; $s = \max\{p, q\} = 10$; $2s = 20$.

The *10*-even-edge-graceful labeling of C_5^+ is given in Figure 11.

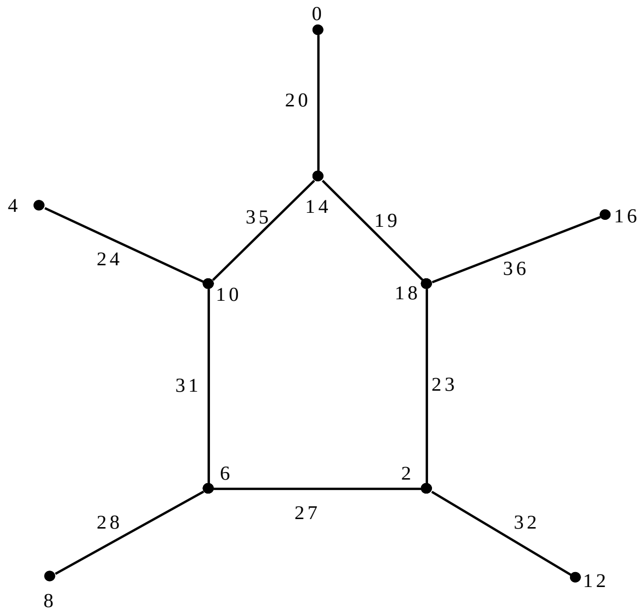

Figure 11: 10-*EEGL* of C_5^+

Definition 2.10

A graph $H_m(G)$ is obtained from a graph G by replacing each edge with m parallel edges.

Theorem 2.11

The graph $H_n(P_n)$, ($n \geq 2$) of even order is k-even-edge-graceful for all $k \equiv z \pmod{p - 1}$, where $0 \leq z \leq p - 2$ and n is even.

Proof

Let $\{v_1, v_2, ..., v_n\}$ be the vertices of $H_n(P_n)$. Let the edges e_{ij} of $H_n(P_n)$ be defined by

$e_{ij} = (v_i, v_{i+1})$ for $1 \leq i \leq n - 1$, $1 \leq j \leq n$ (see Figure 12).

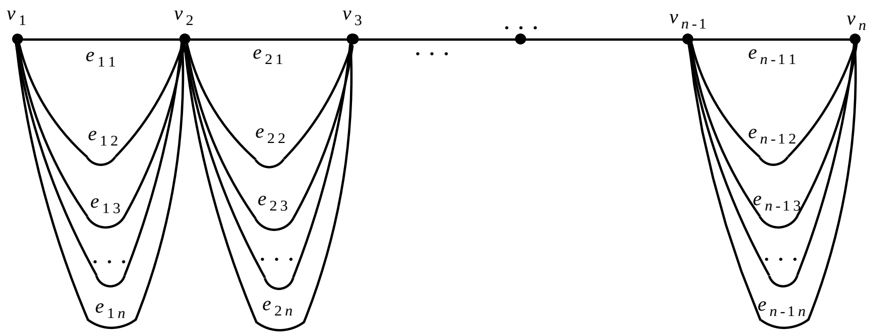

Figure 12: $H_n(P_n)$ with ordinary labeling

For this graph, $p = n$; $q = n(n-1)$.
First, we label the edges as follows:
For $k \geq 1$,

$$f(e_{ij}) = \begin{cases} 2k + ni + 2j - 5 & \text{for } 1 \leq i \leq n-1,\ i \text{ is odd and } 1 \leq j \leq n \\ 2k + 2j + n(i-1) - 4 & \text{for } 1 \leq i \leq n-1,\ i \text{ is even and } 1 \leq j \leq n. \end{cases}$$

Then the induced vertex labels are as follows:

Case 1: $k \equiv 0 \pmod{p-1}$

$f^+(v_1) = 2n(n-2)$.

$$f^+(v_i) = \begin{cases} n(2n + 2i - 9) & \text{for } 2 \leq i < 4 \\ n(2i - 7) & \text{for } 4 \leq i \leq n-1. \end{cases}$$

$f^+(v_n) = n(n-4)$.

Case 2: $k \equiv z \pmod{p-1}$, $1 \leq z \leq p-2$

$f^+(v_1) = 2n(z-1)$.

Subcase (i): $k \equiv 1 \pmod{p-1}$

$f^+(v_i) = n(2i-3)$ for $2 \leq i \leq n-1$.

$f^+(v_n) = n[n + 2(z-2)]$.

Subcase (ii): $k \equiv z \pmod{p-1}$, $2 \leq z \leq \frac{p}{2}$

$$f^+(v_i) = \begin{cases} n(2i-3) + 4n(z-1) & \text{for } 2 \leq i < n - 2z + 3 \\ n[2(i + 2z - n) - 5] & \text{for } n - 2z + 3 \leq i \leq n-1. \end{cases}$$

$f^+(v_n) = n[n + 2(z-2)]$.

Subcase (iii): $k \equiv z \pmod{p-1}$, $\frac{p+2}{2} \leq z \leq p-2$

$$f^+(v_i) = \begin{cases} n[2(i + 2z - n) - 5] & \text{for } 2 \leq i < 2n - 2z + 2 \\ n[2(i - 2n + 2z) - 3] & \text{for } 2n - 2z + 2 \leq i \leq n-1. \end{cases}$$

$f^+(v_n) = n[2(z-1) - n]$.

Therefore, $f^+(V) \subseteq \{0, 2, 4, ..., 2s-2\}$, where $s = \max\{p, q\} = n(n-1)$. So, it follows that the vertex labels are all distinct and even. Hence, the graph $H_n(P_n)$, $(n \geq 2)$ of even order is k-even-edge-graceful for all $k \equiv z \pmod{p-1}$, where $0 \leq z \leq p-2$ and n is even. ■

For example, consider the graph $H_8(P_8)$. Here $p = 8$; $q = 56$;

$s = \max\{p, q\} = 56$; $2s = 112$. The *6-EEGL* of $H_8(P_8)$ is given in Figure 13.

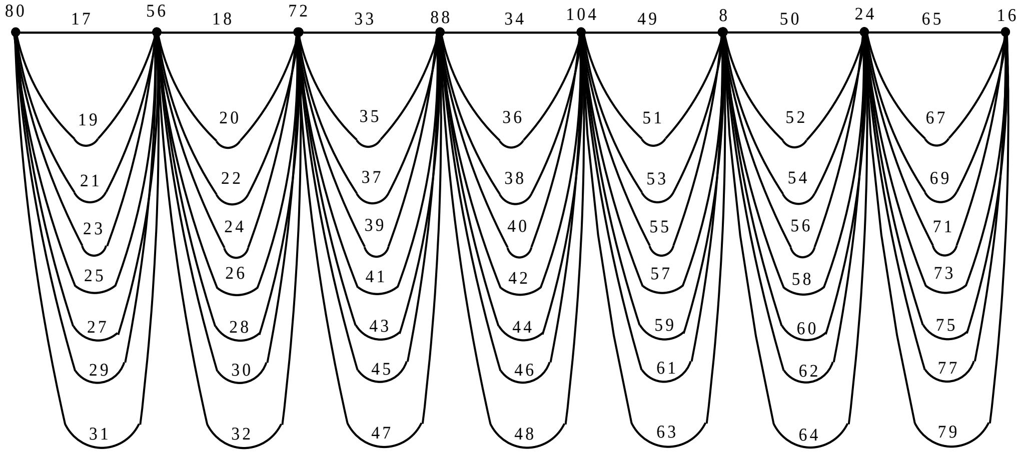

Figure 13: 6-*EEGL* of $H_8(P_8)$

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